On Foucault's Pendulum.

By William Duncan MacMillan.

§ 1. Introduction.

A number of papers have appeared during the past sixty years on the theory of the motion of the Foucault pendulum,[*] but the theory is still far from being in a satisfactory state. The theory given in the various treatises on mechanics includes only the case of infinitesimal oscillations. For oscillations of this type the equations of motion are completely integrable, and it is found that, if the motion is referred to a horizontal plane rotating in clockwise direction with uniform angular speed of period $\dfrac{24}{\sin\beta}$ hours, where β is the latitude of the place, the pendulum describes a relatively long, narrow ellipse in which the ratio of the minor to the major axis is the same as the ratio of the period of a single oscillation of the pendulum to the period of the rotating plane. This result is independent of the azimuth of the initial vertical plane of the pendulum, so that the theory for this case is complete.

For finite oscillations, however, only approximate solutions have been given. If ω represents the angular rate of the earth's rotation, and if terms of the order ω^2 and higher are neglected in convenient places, the equations of motion can be integrated by means of elliptic functions.[†] Since the quantity ω^2 is very small, the results obtained by this process doubtless represent the motion very accurately for a considerable interval of time; but as the results obtained do not satisfy the equations of motion, no inferences can be drawn from them with safety over extended intervals of time.

It is the purpose of the present paper to set forth explicitly two rigorous particular solutions of the equations of motion as they are usually given.[‡] In these equations the oblateness of the earth and the terms in ω^2 are neglected. It will be shown that if the pendulum is started from rest in the plane of the meridian, if certain conditions of commensurability are satisfied, and if the

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[*] References to the literature of this subject are given in the " Encyklopädie der Mathematischen Wissenschaften," IV, 7, S. 6.

[†] Chessin, American Journal of Mathematics, Vol. XVII, 1895.

[‡] Poisson, *Journal Polytechnique*, 1838. Routh, "Advanced Rigid Dynamics," p. 27. Appel, "Traité de Mécanique Rationelle," II, p. 310, edition of 1896.

oscillations of the pendulum are not too large, then the motion of the pendulum is always periodic. Furthermore, it will be shown that the period of the rotating plane, which is 24 hours divided by the sine of latitude for infinitesimal oscillations, increases as the amplitude of the oscillations of the pendulum increases. Similar results are obtained when the pendulum is started into motion with an initial impulse.

§ 2. *The Differential Equations.*

It is assumed that the earth is a sphere and that the xy-plane is tangent to the sphere at the place of observation. The positive end of the x-axis is towards the south, the positive end of the y-axis is towards the east, and the positive end of the z-axis is directed downward. The origin is taken at the point of suspension of the pendulum. We will let ω denote the rate of the earth's rotation, β the latitude, l the length and mT the tension of the suspending wire, and g the acceleration of gravity. The equations of motion are

$$\left.\begin{array}{l} \dfrac{d^2x}{dt^2} = -\dfrac{T}{l}\,x + 2\omega\,\sin\,\beta\,\dfrac{dy}{dt}\,, \\[2mm] \dfrac{d^2y}{dt^2} = -\dfrac{T}{l}\,y - 2\omega\,\sin\,\beta\,\dfrac{dx}{dt} + 2\omega\,\cos\,\beta\,\dfrac{dz}{dt}\,, \\[2mm] \dfrac{d^2z}{dt^2} = -\dfrac{T}{l}\,z + g - 2\omega\,\cos\,\beta\,\dfrac{dy}{dt}\,. \end{array}\right\} \tag{1}$$

Since the pendulum is of invariable length l, the following relations are always satisfied:

$$\left.\begin{array}{l} x^2 + y^2 + z^2 = l^2\,, \\[2mm] x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0\,, \\[2mm] x\,\dfrac{d^2x}{dt^2} + y\,\dfrac{d^2y}{dt^2} + z\,\dfrac{d^2z}{dt^2} = -\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2\right]; \end{array}\right\} \tag{2}$$

and the energy integral is

$$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2 = 2gl\left(\dfrac{z}{l} - c\right), \tag{3}$$

where c is the constant of integration. Combining (1), (2) and (3), it is found without difficulty that

$$\dfrac{T}{l} = \dfrac{g}{l}\left(3\,\dfrac{z}{l} - 2c\right) + \dfrac{2\omega}{l^2}\left[\left(x\,\dfrac{dy}{dt} - y\,\dfrac{dx}{dt}\right)\sin\,\beta + \left(y\,\dfrac{dz}{dt} - z\,\dfrac{dy}{dt}\right)\cos\,\beta\right]. \tag{4}$$

It will be supposed that the pendulum is initially displaced towards the south through an angle whose sine is μ. Then the initial values are

$$x=l\mu, \quad y=0, \quad z=l\sqrt{1-\mu^2}, \quad \frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0, \tag{5}$$

from which it follows that $c=\sqrt{1-\mu^2}$.

Let us now make a change of variables by the transformation

$$\left.\begin{array}{l} x=l[\xi \cos nt+\eta \sin nt], \\ y=l[-\xi \sin nt+\eta \cos nt], \\ z=l[1-\zeta], \end{array}\right\} \tag{6}$$

so that $\xi^2+\eta^2+(1-\zeta)^2=1$, and for brevity let us take $\omega \sin \beta=\sigma$, $\omega \cos \beta=\sigma_1$. Then the motion of the pendulum is referred to a system of rectangular axes rotating with the angular velocity n, where n is an arbitrary at our disposal, and the differential equations are

$$\left.\begin{array}{l} \dfrac{d^2\xi}{dt^2}+\left[\dfrac{T}{l}+n^2-2n(n-\sigma)\right]\xi=-2(n-\sigma)\dfrac{d\eta}{dt}+2\sigma_1 \sin nt\dfrac{d\zeta}{dt}, \\[2ex] \dfrac{d^2\eta}{dt^2}+\left[\dfrac{T}{l}+n^2-2n(n-\sigma)\right]\eta=+2(n-\sigma)\dfrac{d\xi}{dt}-2\sigma_1 \cos nt\dfrac{d\zeta}{dt}, \\[2ex] \dfrac{d^2\zeta}{dt^2}+\dfrac{T}{l}\zeta=\dfrac{T-g}{l}-2\sigma_1\left(n\xi-\dfrac{d\eta}{dt}\right)\cos nt-2\sigma_1\left(\dfrac{d\xi}{dt}+n\eta\right)\sin nt. \end{array}\right\} \tag{7}$$

From the relation $\zeta=1-\sqrt{1-(\xi^2+\eta^2)}$ it is seen that, if ξ and η are small quantities of the first order, ζ is a small quantity of the second order. Neglecting terms of the second and higher orders and choosing $n=\sigma$, the differential equations become simply

$$\left.\begin{array}{l} \dfrac{d^2\xi}{dt^2}+\left[\dfrac{g}{l}+\sigma^2\right]\xi=0, \\[2ex] \dfrac{d^2\eta}{dt^2}+\left[\dfrac{g}{l}+\sigma^2\right]\eta=0. \end{array}\right\} \tag{8}$$

In order to satisfy the initial conditions in the xy-plane we must have at $t=0$, $\xi=\mu$, $\dfrac{d\xi}{dt}=0$, $\eta=0$, $\dfrac{d\eta}{dt}=\sigma\mu$, and therefore the solution is

$$\xi=\mu \cos\sqrt{\dfrac{g}{l}+\sigma^2}\,t, \quad \eta=\dfrac{\sigma\mu}{\sqrt{\dfrac{g}{l}+\sigma^2}} \sin \sqrt{\dfrac{g}{l}+\sigma^2}\,t, \tag{9}$$

which is an ellipse of which the semi-major axis is μ and the semi-minor axis is $\dfrac{\sigma\mu}{\sqrt{\dfrac{g}{l}+\sigma^2}}$. Hence the theorem of Chevilliet for infinitesimal oscillations,

"*The ratio of the major to the minor axis is equal to the ratio of the period of the rotating plane to the period of oscillation in the ellipse.*"

There is no difficulty in introducing an azimuth constant into this solution if it is so desired.

§ 3. *Development of a Periodic Solution.*

The choice $n = \sigma$ certainly simplifies the differential equations. Nevertheless it does not seem to be convenient to develop a periodic solution in this manner. For finite oscillations we will take $n = \dfrac{\sigma}{1+x}$, where x is an arbitrary constant, the form and value of which will be determined by the initial and periodicity conditions. It will be convenient also to take $l = \dfrac{l_0}{1+\lambda}$, where l_0 is a fixed constant and λ is a constant of the same type as x. We will transform the independent variable by taking $t = (1+x)\tau$ so that $nt = \sigma\tau$, and, finally, we will write $\dfrac{g}{l_0} + \sigma^2 = p^2$. Since the length of the pendulum is invariable throughout the motion, we can discard the differential equation in ζ, and there remains

$$\left.\begin{array}{l} \xi'' + \left[(1+x)^2 \dfrac{T'}{l} + (1+2x)\sigma^2 \right]\xi = +2x\sigma\eta' + 2(1+x)\sigma_1\zeta' \sin \sigma\tau, \\[3mm] \eta'' + \left[(1+x)^2 \dfrac{T}{l} + (1+2x)\sigma^2 \right]\eta = -2x\sigma\xi' - 2(1+x)\sigma_1\zeta' \cos \sigma\tau, \end{array}\right\} \quad (10)$$

where accents denote derivatives with respect to τ, and

$$(1+x)^2 \frac{T}{l} = (1+x)^2(1+\lambda)\frac{g}{l_0} \left[3\sqrt{1-(\xi^2+\eta^2)} - 2\sqrt{1-\mu^2}\right]$$

$$+ 2(1+x)\sigma(\xi\eta' - \eta\xi') - 2\sigma^2(1+x)(\xi^2+\eta^2)$$

$$+ 2(1+x)\sigma_1\left[\left((\sigma\xi - \eta')\sqrt{1-(\xi^2+\eta^2)} - \frac{\eta(\xi\xi'+\eta\eta')}{\sqrt{1-(\xi^2+\eta^2)}}\right)\cos \sigma\tau\right.$$

$$\left. + \left((\sigma\eta + \xi')\sqrt{1-(\xi^2+\eta^2)} + \frac{\xi(\xi\xi'+\eta\eta')}{\sqrt{1-(\xi^2+\eta^2)}}\right)\sin \sigma\tau\right], \quad (11)$$

and $\zeta = 1 - \sqrt{1-(\xi^2+\eta^2)}$. The integral becomes

$$\xi'^2 + \eta'^2 + \frac{(\xi\xi'+\eta\eta')^2}{1-(\xi^2+\eta^2)} = 2\sigma(\xi\eta' - \eta\xi') - \sigma^2(\xi^2+\eta^2)$$

$$+ \frac{2g}{l_0}(1+\lambda)(1+x)^2\left[\sqrt{1-(\xi^2+\eta^2)} - \sqrt{1-\mu^2}\right]. \quad (12)$$

We propose to integrate these equations as power series in μ, and we therefore assume

$$\xi = \xi_1\mu + \xi_2\mu^2 + \xi_3\mu^3 + \ldots,$$
$$\eta = \eta_1\mu + \eta_2\mu^2 + \eta_3\mu^3 + \ldots,$$
$$x = x_1\mu + x_2\mu^2 + x_3\mu^3 + \ldots,$$
$$\lambda = \lambda_1\mu + \lambda_2\mu^2 + \lambda_3\mu^3 + \ldots$$

In these series the constants x_i and λ_i will be chosen so that the coefficients ξ_j, η_j shall be periodic functions of τ. As is seen from equations (8), the free period of the differential equations is $\dfrac{2\pi}{p}$ and the forced period $\dfrac{2\pi}{\sigma}$. It is necessary to assume, therefore, that the constant l_0 has such a value that $p = \sqrt{\dfrac{g}{l_0} + \sigma^2}$ is commensurable with σ. This assumption is of importance in the theory of convergence of the series, but, obviously, is of little importance physically. It is found readily by a few preliminary computations that $x_1 = \lambda_1 = \sigma\eta_1 - \xi_1' = \xi_2 = 0$, and $\xi_1\eta_1' - \eta_1\xi_1' = \sigma$. Utilizing these facts to simplify the expansions, the differential equations become

$$\sum_{j=1}^{\infty} [\xi_j'' + p^2\xi_j]\mu^j =$$
$$[-2\sigma_1(\sigma\eta_1 + \xi_1')\xi_1 \sin \sigma\tau + \sigma_1(\xi_1^2 + \eta_1^2)' \sin \sigma\tau]\mu^2$$
$$+ [-((p^2 - \sigma^2)\lambda_2 + 2p^2x_2 + p^2 + \sigma^2)\xi_1 + 2x_2\sigma\eta_1' + (\tfrac{3}{2}p^2 + \tfrac{1}{2}\sigma^2)(\xi_1^2 + \eta_1^2)\xi_1$$
$$+ 2\sigma_1\xi_1\eta_2' \cos \sigma\tau + (2\sigma_1(\eta_1\eta_2)' - 2\sigma\sigma_1\xi_1\eta_2) \sin \sigma\tau]\mu^3$$
$$+ [-((p^2 - \sigma^2)\lambda_3 + 2p^2x_3)\xi_1 + 2x_3\sigma\eta_1' + (3p^2 + \sigma^2)\xi_1\eta_1\eta_2 - 2\sigma\xi_1(\xi_1\eta_2' - \xi_2\eta_1') + 2x_2\sigma\eta_2'$$
$$+ \{-2\sigma_1\xi_1(\sigma\xi_3 - \eta_3') + \sigma_1\xi_1\eta_1(\xi_1^2 + \eta_1^2)'\}\cos \sigma\tau + \{-2\sigma_1x_2\xi_1(\sigma\eta_1 + \xi_1')$$
$$- 2\sigma_1\xi_1(\sigma\eta_3 + \xi_3') + \sigma\xi_1(\sigma\eta_1 + \xi_1')(\xi_1^2 + \eta_1^2) - \sigma\xi_1^2(\xi_1^2 + \eta_1^2)' - 2\sigma_1\xi_3(\sigma\eta_1 + \xi_1')$$
$$+ 2\sigma_1(\xi_1\xi_3 + \eta_1\eta_3)' + 2\sigma_1\eta_2\eta_2' + \tfrac{1}{2}\sigma_1(\xi_1^2 + \eta_1^2)(\xi_1^2 + \eta_1^2)'$$
$$+ x_2\sigma_1(\xi_1^2 + \eta_1^2)'\} \sin \sigma\tau]\mu^4 + \ldots, \tag{13}$$

$$\sum_{j=1}^{\infty} [\eta_j'' + p^2\eta_j]\mu^j =$$
$$[-2\sigma_1(\sigma\eta_1 + \xi_1')\eta_1 \sin \sigma\tau - \sigma_1(\xi_1^2 + \eta_1^2)' \cos \sigma\tau]\mu^2$$
$$+ [-(p^2 - \sigma^2)\lambda_2\eta_1 - 2x_2(p^2\eta_1 + \sigma\xi_1') - (p^2 + \sigma^2)\eta_1 + \tfrac{1}{2}(3p^2 + \sigma^2)(\xi_1^2 + \eta_1^2)\eta_1$$
$$+ 2\sigma_1\eta_1(\eta_2 \cos \sigma\tau)' - 2\sigma_1(\eta_1\eta_2)' \cos \sigma\tau - 2\sigma_1(\sigma\eta_1 + \xi_1')\eta_2 \sin \sigma\tau]\mu^3$$
$$+ [-(p^2 - \sigma^2)\lambda_3\eta_1 - 2x_3(p^2\eta_1 + \sigma\xi_1') + (3p^2 + \sigma^2)\eta_1^2\eta_2 - 2\sigma\eta_1(\xi_1\eta_2' - \xi_1'\eta_2)$$
$$- \{(p^2 - \sigma^2)\lambda_2 + 2p^2x_2 + p^2 + \sigma^2\}\eta_2 + \tfrac{1}{2}(3p^2 + \sigma^2)(\xi_1^2 + \eta_1^2)\eta_2 + 2\sigma_1\eta_2(\eta_2 \cos \sigma\tau)'$$
$$+ \{-2\sigma_1\eta_1(\sigma\xi_3 - \eta_3') + \sigma_1\eta_1^2(\xi_1^2 + \eta_1^2)' - 2\sigma_1(\xi_1\xi_3 + \eta_1\eta_3)' - 2\sigma_1\eta_2\eta_2'$$
$$- \tfrac{1}{2}\sigma_1(\xi_1^2 + \eta_1^2)(\xi_1^2 + \eta_1^2)' - \sigma_1x_2(\xi_1^2 + \eta_1^2)'\} \cos \sigma\tau + \{-2\sigma_1x_2\eta_1(\sigma\eta_1 + \xi_1')$$
$$- 2\sigma_1\eta_1(\sigma\eta_3 + \xi_3') + \sigma_1\eta_1(\sigma\eta_1 + \xi_1')(\xi_1^2 + \eta_1^2) - \sigma\eta_1\xi_1(\xi_1^2 + \eta_1^2)'$$
$$- 2\sigma_1\eta_3(\sigma\eta_1 + \xi_1')\} \sin \sigma\tau]\mu^4 + \ldots \tag{14}$$

From the initial conditions (5) and the equations of transformation (6), it is found that the initial values of ξ and η are

$$\xi(0) = \mu, \quad \eta(0) = 0; \quad \xi'(0) = 0, \quad \eta'(0) = \sigma\mu.$$

Consequently,

$$\left.\begin{array}{ll} \xi_1(0) = 1, & \eta_1(0) = 0; \quad \xi_1'(0) = 0, \quad \eta_1'(0) = \sigma; \\ \xi_j(0) = 0, & \eta_j(0) = 0; \quad \xi_j'(0) = 0, \quad \eta_j'(0) = 0, \quad j = 2, \ldots, \infty. \end{array}\right\} \quad (15)$$

Equating the coefficients of the first powers of μ in the left and right members of (13) and (14), we find

$$\left.\begin{array}{l} \xi_1'' + p^2\xi_1 = 0, \\ \eta_1'' + p^2\eta_1 = 0, \end{array}\right\} \quad (16)$$

and the solutions of these equations which satisfy the initial conditions are

$$\xi_1 = \cos p\tau, \quad \eta_1 = \frac{\sigma}{p} \sin p\tau. \Big\} \quad (17)$$

From these expressions it results that

$$\xi_1^2 + \eta_1^2 = \frac{p^2+\sigma^2}{2p^2} + \frac{p^2-\sigma^2}{2p^2} \cos 2p\tau, \quad \sigma\eta_1 + \xi_1' = -\frac{p^2-\sigma^2}{p} \sin p\tau, \quad \sigma\xi_1 - \eta_1' = 0. \quad (18)$$

From the coefficients of the second power of μ in (13) and (14), and from the values given in (17) and (18), it is found that

$$\left.\begin{array}{l} \xi_2'' + p^2\xi_2 = 0, \\ \eta_2'' + p^2\eta_2 = \dfrac{(p^2-\sigma^2)\sigma\sigma_1}{p^2} \sin \sigma\tau + \dfrac{(p+\sigma)^2(p-\sigma)\sigma_1}{2p^2} \sin (2p-\sigma)\tau \\ \qquad\qquad\qquad + \dfrac{(p-\sigma)^2(p+\sigma)\sigma_1}{2p^2} \sin (2p+\sigma)\tau, \end{array}\right\} \quad (19)$$

On integrating these equations and imposing the initial conditions, we find

$$\left.\begin{array}{l} \xi_2 = 0, \\ \eta_2 = \dfrac{6(p^2-\sigma^2)\sigma_1}{p(9p^2-\sigma^2)} \sin p\tau + \dfrac{\sigma\sigma_1}{p^2} \sin \sigma\tau - \dfrac{(p-\sigma)^2\sigma_1}{2p^2(3p+\sigma)} \sin(2p+\sigma)\tau \\ \qquad\qquad\qquad - \dfrac{(p+\sigma)^2\sigma_1}{2p^2(3p-\sigma)} \sin(2p-\sigma)\tau. \end{array}\right\} \quad (20)$$

Similarly, the differential equations obtained from the coefficients of the third power of μ in (13) and (14) are

$$\xi_3'' + p^2\xi_3 =$$

$$\left[-(\lambda_2 + 2\varkappa_2) + \frac{p^2-\sigma^2}{8p^2} - \frac{(3p^2+\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)} \right](p^2-\sigma^2)\cos p\tau$$

$$+ \left[\frac{3p^2+\sigma^2}{8p^2} - \frac{(3p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)} \right](p^2-\sigma^2)\cos 3p\tau + \frac{6(p^2-\sigma^2)\sigma_1^2}{9p^2-\sigma^2}\cos \sigma\tau$$

$$+ \frac{3(p^2-\sigma^2)(p-\sigma)\sigma_1^2}{p(9p^2-\sigma^2)}\cos(2p+\sigma)\tau + \frac{3(p^2-\sigma^2)(p+\sigma)\sigma_1^2}{p(9p^2-\sigma^2)}\cos(2p-\sigma)\tau$$

$$+ \frac{(-2p^4+p^3\sigma+9p^2\sigma^2+5p\sigma^3-3\sigma^4)\sigma_1^2}{4p^3(3p+\sigma)}\cos(p+2\sigma)\tau$$

$$+ \frac{(-2p^4-p^3\sigma+9p^2\sigma^2-5p\sigma^3-3\sigma^4)\sigma_1^2}{4p^3(3p-\sigma)}\cos(p-2\sigma)\tau$$

$$- \frac{(p-\sigma)^3(2p+\sigma)\sigma_1^2}{4p^3(3p+\sigma)}\cos(3p+2\sigma)\tau - \frac{(p+\sigma)^3(2p-\sigma)\sigma_1^2}{4p^3(3p-\sigma)}\cos(3p-2\sigma)\tau, \quad (21)$$

$$\eta_3'' + p^2\eta_3 =$$

$$\left[-\lambda_2 - \frac{5p^2+3\sigma^2}{8p^2} + \frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)} \right]\frac{\sigma}{p}(p^2-\sigma^2)\sin p\tau$$

$$+ \left[\frac{3p^2+\sigma^2}{8p^2} + \frac{(5p^2+\sigma^2)\sigma_1^2}{p(9p^2-\sigma^2)} \right]\frac{\sigma}{p}(p^2-\sigma^2)\sin 3p\tau + \frac{6(p^2-2\sigma^2)(p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin \sigma\tau$$

$$- \frac{3(p^2-\sigma^2)(p+2\sigma)(p-\sigma)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin(2p+\sigma)\tau$$

$$+ \frac{3(p^2-\sigma^2)(p-2\sigma)(p+\sigma)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin(2p-\sigma)\tau$$

$$- \frac{(p^4+3p^3\sigma+9p^2\sigma^2-7p\sigma^3-6\sigma^4)\sigma_1^2}{4p^3(3p+\sigma)}\sin(p+2\sigma)\tau$$

$$+ \frac{(p^4-3p^3\sigma+9p^2\sigma^2+7p\sigma^3-6\sigma^4)\sigma_1^2}{4p^3(3p-\sigma)}\sin(p-2\sigma)\tau$$

$$+ \frac{(p-\sigma)^3(p+2\sigma)\sigma_1^2}{4p^3(3p+\sigma)}\sin(3p+2\sigma)\tau - \frac{(p+\sigma)^3(p-2\sigma)\sigma_1^2}{4p^3(3p-\sigma)}\sin(3p-2\sigma)\tau. \quad (22)$$

In order that the solution of these equations may be periodic, it is necessary and sufficient that the coefficient of cos $p\tau$ in (21) and the coefficient of sin $p\tau$ in (22) should vanish. Both of these coefficients carry $(p^2-\sigma^2)$ as a factor, but this factor cannot vanish, since σ is very small as compared with p. The coefficient of sin $p\tau$ in (22) carries σ/p as a factor, and this can vanish only at the earth's equator. For other places on the earth's surface we must have

$$\lambda_2 + 2\varkappa_2 = + \frac{p^2-\sigma^2}{8p^2} - \frac{(3p^2+\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)},$$

$$\lambda_2 = - \frac{5p^2+3\sigma^2}{8p^2} + \frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}.$$

From these equations it results that

$$\left.\begin{aligned}
\lambda_2 &= -\frac{5p^2+3\sigma^2}{8p^2} + \frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}, \\
\varkappa_2 &= +\frac{3p^2+\sigma^2}{8p^2} - \frac{(5p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}.
\end{aligned}\right\} \qquad (23)$$

Since σ and σ_1 are very small as compared with p, it is seen that λ_2 is approximately equal to $-\tfrac{5}{8}$, and $\varkappa_2$ is approximately equal to $+\tfrac{3}{8}$.

Using the values of λ_2 and $\varkappa_2$ given in (23), the solutions of (21) and (22) which satisfy the initial conditions are

$$\begin{aligned}
\xi_3 =& \left[\frac{(3p^2+\sigma^2)(p^2-\sigma^2)}{64p^4} - \frac{(243p^8-256p^6\sigma^2+208p^4\sigma^4-4p^2\sigma^6+\sigma^8)\sigma_1^2}{8p^4(p^2-\sigma^2)(9p^2-\sigma^2)^2}\right]\cos p\tau \\
&+ \left[-\frac{(3p^2+\sigma^2)(p^2-\sigma^2)}{64p^4} + \frac{(3p^2-\sigma^2)(p^2-\sigma^2)\sigma_1^2}{8p^4(9p^2-\sigma^2)}\right]\cos 3p\tau + \frac{6\sigma_1^2}{9p^2-\sigma^2}\cos \sigma\tau \\
&- \frac{3(p^2-\sigma^2)(p-\sigma)\sigma_1^2}{p(p+\sigma)(3p+\sigma)(9p^2-\sigma^2)}\cos (2p+\sigma)\tau \\
&- \frac{3(p^2-\sigma^2)(p+\sigma)\sigma_1^2}{p(p-\sigma)(3p-\sigma)(9p^2-\sigma^2)}\cos (2p-\sigma)\tau \\
&+ \frac{(2p^4-p^3\sigma-9p^2\sigma^2-5p\sigma^3+3\sigma^4)\sigma_1^2}{16p^3\sigma(p+\sigma)(3p+\sigma)}\cos (p+2\sigma)\tau \\
&- \frac{(2p^4+p^3\sigma-9p^2\sigma^2+5p\sigma^3+3\sigma^4)\sigma_1^2}{16p^3\sigma(p-\sigma)(3p-\sigma)}\cos (p-2\sigma)\tau \\
&+ \frac{(p-\sigma)^3\sigma_1^2}{16p^3(p+\sigma)(3p+\sigma)}\cos (3p+2\sigma)\tau + \frac{(p+\sigma)^3\sigma_1^2}{16p^3(p-\sigma)(3p-\sigma)}\cos (3p-2\sigma)\tau, \quad (24)
\end{aligned}$$

$$\begin{aligned}
\eta_3 =& \left[\frac{3(3p^2+\sigma^2)(p^2-\sigma^2)\sigma}{64p^5} - \frac{(27p^8+900p^6\sigma^2-572p^4\sigma^4+32p^2\sigma^6-3\sigma^8)\sigma_1^2}{8p^2\sigma(9p^2-\sigma^2)^2}\right]\sin p\tau \\
&- \left[\frac{(3p^2+\sigma^2)(p^2-\sigma^2)\sigma}{64p^5} + \frac{(5p^2+\sigma^2)(p^2-\sigma^2)\sigma\sigma_1^2}{8p^5(9p^2-\sigma^2)}\right]\sin 3p\tau + \frac{6(p^2-2\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin \sigma\tau \\
&+ \frac{3(p^2-\sigma^2)(p-\sigma)(p+2\sigma)\sigma_1^2}{p^2(p+\sigma)(3p+\sigma)(9p^2-\sigma^2)}\sin (2p+\sigma)\tau \\
&- \frac{3(p^2-\sigma^2)(p+\sigma)(p-2\sigma)\sigma_1^2}{p^2(p-\sigma)(3p-\sigma)(9p^2-\sigma^2)}\sin (2p-\sigma)\tau \\
&+ \frac{(p^4+3p^3\sigma+9p^2\sigma^2-7p\sigma^3-6\sigma^4)\sigma_1^2}{16p^3\sigma(p+\sigma)(3p+\sigma)}\sin (p+2\sigma)\tau \\
&+ \frac{(p^4-3p^3\sigma+9p^2\sigma^2+7p\sigma^3-6\sigma^4)\sigma_1^2}{16p^3\sigma(p-\sigma)(3p-\sigma)}\sin (p-2\sigma)\tau \\
&- \frac{(p-\sigma)^3\sigma_1^2}{16p^3(p+\sigma)(3p+\sigma)}\sin (3p+2\sigma)\tau + \frac{(p+\sigma)^3\sigma_1^2}{16p^3(p-\sigma)(3p-\sigma)}\sin (3p-2\sigma)\tau. \quad (25)
\end{aligned}$$

§ 4. *Properties of the Solution.*

It is not necessary to carry the computation any further, though by induction it will be shown that it is possible to carry it as far as is desired, and that the constants λ_{i-1} and $\varkappa_{i-1}$ can always be determined in the coefficient of μ^i so as to keep the solution periodic. So far as it has been computed, ξ is a cosine series and η is a sine series. Let us suppose that this property holds up to and including the coefficients of μ^{i-1}. Then from (11) and from the properties of evenness and oddness it is seen that T/l is a cosine series up to and including the coefficient of μ^{i-1}. Then it follows from (10) that the differential equations for ξ_i and η_i have the form

$$\left.\begin{aligned}
\xi''_i + p_2^2\xi_i &= [-(\lambda_{i-1}+2\varkappa_{i-1})\,(p^2-\sigma^2) + A_i]\cos p\tau + \text{other cosine terms,} \\
\eta''_i + p^2\eta_i &= \left[-\lambda_{i-1}\frac{\sigma}{p}\,(p^2-\sigma^2) + B_i\right]\sin p\tau + \text{other sine terms,}
\end{aligned}\right\} \quad (26)$$

where A_i and B_i are known constants. In order that the solution may be periodic, we must have

$$\lambda_{i-1}+2\varkappa_{i-1} = \frac{A_i}{p^2-\sigma^2}\,,$$

$$\lambda_{i-1} = \frac{B_i p}{\sigma\,(p^2-\sigma^2)}\,;$$

and these equations uniquely determine λ_{i-1} and $\varkappa_{i-1}$. The solution of (26) which satisfies the initial conditions is then

$$\xi_i = \text{sum of cosine terms,}$$
$$\eta_i = \text{sum of sine terms.}$$

Thus ξ_i and η_i have the properties which were assumed for ξ_{i-1} and η_{i-1}. The property holds, therefore, for every i, and consequently ξ is a periodic cosine series and η is a periodic sine series.

We have therefore formally determined a periodic solution of the equations of motion, and we are assured of its convergence by the general existence theorem of periodic solutions.*

A second property of the solution which has just been obtained is as follows: If $\genfrac{}{}{0pt}{}{\sin}{\cos}\,(ip\pm j\sigma)\tau$ is a term in the solution, then $i+j$ is an odd integer. This is certainly true up to and including the coefficients of μ^3. That it is general is readily seen by induction. Bearing in mind the properties of even and odd multiples, it is readily seen from (11) that if $i+j$ is an odd integer in all coeffi-

cients of ξ and η up to and including $\mu^{\kappa-1}$, then the expansion of T/l contains only terms in which $i+j$ is an even integer; and consequently from (10) it is seen that ξ_κ and η_κ contain only terms in which $i+j$ is odd. The property is therefore general.

We have already supposed that p and σ are commensurable numbers. Let us take, then, $p=m\phi$ and $\sigma=n\phi$, where m and n are integers relatively prime. Any term in the solution, $\genfrac{}{}{0pt}{}{\cos}{\sin}(ip\pm j\sigma)\tau$, can therefore be written $\genfrac{}{}{0pt}{}{\cos}{\sin}(im\pm jn)\phi\tau$. If m and n are both odd, then $(im\pm jn)$ is necessarily odd and the solution contains only odd multiples of $\phi\tau$. The projection of the path described by the pendulum upon the rotating $\xi\eta$-plane is therefore symmetrical with respect to both the ξ-axis and the η-axis. In the fixed xy-plane, however, since

$$x=l[\xi \cos \sigma\tau+\eta \sin \sigma\tau],$$
$$y=l[-\xi \sin \sigma\tau+\eta \cos \sigma\tau],$$

the expressions for x and y will contain only even multiples of $\phi\tau$. Therefore the xy-curve described in the interval $\dfrac{\pi}{\phi}\leqq\tau\leqq\dfrac{2\pi}{\phi}$ is identical with that described in the interval $0\leqq\tau\leqq\dfrac{\pi}{\phi}$, though in the rotating plane they are distinct. Since x is a cosine series and y is a sine series, the orbit is symmetrical with respect to the x-axis, and it has a cusp at the point for which $\tau=0$. The pendulum is again at this cusp when $\tau=\dfrac{\pi}{\phi}$. Since the orbit is symmetrical with respect to the x-axis, it follows that at $\tau=\dfrac{\pi}{2\phi}$ the orbit again crosses the x-axis perpendicularly or at a cusp.

If m and n are not both odd (they cannot both be even since they are relatively prime), then the expressions for ξ and η in the rotating plane contain both even and odd multiples of $\phi\tau$, and the orbit, while symmetrical with respect to the ξ-axis, is not symmetrical with respect to the η-axis.

The period of the rotating plane is $\dfrac{2\pi}{n}=\dfrac{2\pi}{\sigma}(1+\varkappa)$. It is seen from (23) that, for small values of μ, $\varkappa$ is approximately $+\tfrac{3}{8}\mu^2$. Consequently the period of the rotating plane increases as the amplitude of the pendulum increases.

The mean period of oscillation of the pendulum is $\dfrac{2\pi}{p}$. In order to find the actual period, which will differ but little from $\dfrac{2\pi}{p}$, we will take $\tau=\dfrac{2n\pi}{p}+\epsilon_n$ and

impose the condition that $r^2 = \xi^2 + \eta^2$ is a maximum, *i. e.*, $\dfrac{dr^2}{d\tau} = 0$. This condition gives the equation

$$0 = \left[-\frac{(p^2 - \sigma^2)}{p} \sin 2p\varepsilon_n \right] \mu^2 + \left[\frac{12(p^2 - \sigma^2)\sigma\sigma_1}{p(9p^2 - \sigma^2)} \sin 2p\varepsilon_n \right.$$
$$\left. - \frac{8(3p^2 + \sigma^2)\sigma\sigma_1}{9p^2 - \sigma^2} \cos 2n\pi \frac{\sigma}{p} + \frac{32\sigma^2\sigma_1}{9p^2 - \sigma^2} \sin 2n\pi \frac{\sigma}{p} \right] \mu^3 + \ldots,$$

from which is obtained

$$\varepsilon_n = \left[\frac{16\sigma^2\sigma_1}{(p^2 - \sigma^2)(9p^2 - \sigma^2)} \sin 2n\pi \frac{\sigma}{p} \right] \mu + \ldots$$

Substituting $\tau = \dfrac{2n\pi}{p} + \varepsilon_n$ in the expressions for ξ and η, we have the coordinates of the apse. or the end of the oscillation. The expression for the η-coordinate of the apse is

$$\eta_a = \left[\frac{\sigma}{p} \sin \left(\frac{32\sigma^2\sigma_1 p}{(p^2 - \sigma^2)(9p^2 - \sigma^2)} \sin 2n\pi \frac{\sigma}{p} \right) \mu \right] \mu + \ldots$$

Since σ and σ_1 each carry ω as a factor, this quantity is of the order ω^4. Thus the line of apsides does not rotate with uniform angular motion, but its departure from uniformity is exceedingly small.

At first thought, it might seem that this solution should reduce at the north pole to that of the simple pendulum referred to rotating axes, and consequently the rotating axes should have the period of 24 hours exactly. On second thought, however, it is seen that at the poles the Foucault pendulum becomes a spherical pendulum referred to rotating axes, for initially the pendulum is at rest with respect to the earth and therefore in motion with respect to fixed axes. Since this motion is counter-clockwise when the pendulum is released, and since in the motion of the spherical pendulum the line of apsides rotates in the same direction as the motion of the pendulum itself, it follows that the line of apsides rotates in the same direction as the earth itself. Therefore it takes the earth more than 24 hours to come back to the same position with respect to the pendulum. The expression for the period of the rotating axes is $P = (1 + x_2\mu^2 + \ldots)24^h$, and the value of x_2 at the pole is positive.

§ 5. *Foucault's Pendulum with Initial Impulse.*

If, however, the pendulum is started from relative rest, in which it hangs freely from its point of suspension, with an initial impulse, the solution so derived should reduce to the simple pendulum at the poles. By exactly the

same method as has been used above, a periodic solution can be obtained, provided the initial impulse be directed towards the east or towards the west.

With the same notation as before, let us suppose that the initial conditions are

$$\xi = \xi' = \eta = 0, \quad \eta' = p\mu,$$

which implies an initial impulse towards the east. The solution is found to be

$$\xi = [0]\mu + \left[\frac{4p^2\sigma\sigma_1}{(p^2-\sigma^2)(9p^2-\sigma^2)} \cos p\tau + \frac{p\sigma_1}{2(p+\sigma)(3p+\sigma)} \cos (2p+\sigma)\tau \right.$$
$$\left. - \frac{p\sigma_1}{2(p-\sigma)(3p-\sigma)} \cos (2p-\sigma)\tau \right]\mu^2 + \ldots,$$

$$\eta = [\sin p\tau]\mu + \left[\frac{4p\sigma^2\sigma_1}{(p^2-\sigma^2)(9p^2-\sigma^2)} \sin p\tau - \frac{\sigma\sigma_1}{(p^2-\sigma^2)} \sin \sigma\tau \right.$$
$$\left. - \frac{\sigma\sigma_1}{2(p+\sigma)(3p+\sigma)} \sin (2p+\sigma)\tau + \frac{\sigma\sigma_1}{2(p-\sigma)(3p-\sigma)} \sin (2p-\sigma)\tau \right]\mu^2 + \ldots,$$

$$\lambda = \left[\frac{p^2+3\sigma^2}{8(p^2-\sigma^2)} - \frac{(11p^2-3\sigma^2)\sigma_1^2}{(p^2-\sigma^2)(9p^2-\sigma^2)} \right]\mu^2 + \ldots,$$

$$\varkappa = \left[\frac{4p^2\sigma_1^2}{(p^2-\sigma^2)(9p^2-\sigma^2)} \right]\mu^2 + \ldots$$

The period of the rotating axes is $P = (1 + \varkappa_2\mu^2 + \ldots) 24^h$; and since $\sigma_1 = \omega \cos \beta$, it is seen that this is 24 hours exactly, at the poles, as it should be.